XI. *A Letter from the Rev.* Patrick Murdocke, *F. R. S. concerning the* mean Motion *of the* Moon's Apogee, *to the Rev. Dr.* Robert Smith, *Master of* Trinity College Cambridge.

Reverend Sir,

Read Jan. 31. 1750.

LAST summer, when I was to pay my respects to you at Trinity College, I gave you some account of the warm dispute, then lately arisen between Mr. de Buffon and Mr. Clairaut, two eminent academicians at Paris; the latter pretending, that the Newtonian law of attraction is inconsistent with the motion of the moon's apogee; and that its quantity ought not to be expressed by $\frac{1}{x^2}$ of the distance, but by two, or perhaps more, terms of a series, as $\frac{1}{x^2} + \frac{a}{x^4}$. Which new doctrine Mr. Clairaut had got inserted in the memoirs of the academy, and Mr. de Buffon had followed him close with another memoir, confuting it.

When I first heard of this controversy, it was impossible to judge of the validity of Mr. Clairaut's reasons, because he kept his *calculus* a profound secret. But an absurd consequence of his new law of attraction occurr'd to me, as soon as Mr. de Buffon mention'd the thing, that, "if we should put the at-" traction, express'd by his two terms, of an assumed "quantity *G*, and resolve the equation, there would

 "necessarily

" necessarily arise two different values of the distance
" x, for the same attractive force."

Suspecting therefore, that some error must have slipt into Mr. Clairaut's reasonings (as he himself afterwards found there had)* I resolved to try, whether, by an arithmetical calculation, from Sir Isaac Newton's propositions only, the motion in question might not be accounted for.

The result of this inquiry I should have taken the liberty to send you before now, but that, other things intervening, I did not think of revising and transcribing it, till lately; that Mr. Walmesley having made me a present of his ingenious treatise on the same subject, it appears, that, however Mr. Clairaut's hypothesis is given up, yet a notion still prevails, as if Sir Isaac Newton's propositions, concerning the motion of *apsids*, were mere mathematical fictions, not applicable to nature.

How far I have succeeded in shewing the contrary, is now submitted to your judgment. And I, at the same time, embrace, with pleasure, an opportunity of professing myself, with the highest respect,

Reverend Sir,

Sradishall, 6 April, 1750.

Your most obliged, and

most obedient humble servant,

Pat. Murdocke.

Of

Of the mean motion of the moon's apogee, according to Sir Isaac Newton.

The rule given by Sir Isaac Newton, in the 9 section of his first book, is to this purpose: Tab. Fig. 3.

1. That, supposing the common law of attraction, and that a central body T attracts the body P, revolving round it in an orbit nearly circular, with a force as unity; if to this be added a constant force, whose ratio to the former is expressed by c; then the angular velocity of the body P, in an immoveable plane, will be to its angular velocity, reckoned from the *apsis* of its orbit, in the subduplicate ratio of $1+c$ to $1+4c$, or as $\sqrt{\frac{1+c}{1+4c}}$ to unity. And therefore, if A represents any arc described by the revolving body in an immoveable plane, $A\times\sqrt{\frac{1+4c}{1+c}}$ will be the corresponding arc in its orbit, reckon'd from the *apsis*. And their difference $A\times\overline{\sqrt{\frac{1+4c}{1+c}}-1}$, will be the gress of the *apsis*.

But if the force of the central body T is diminished by some constant force as c, then the sign of c is changed in these expressions; and the direct motion of the *apsis* will be $A\times\overline{1-\sqrt{\frac{1-4c}{1-c}}}$.

2. And hence, if some foreign variable force, added to, or subtracted from, the central force of attraction, produces a given motion of the *apsis*, retrograde or direct; it is easy to find a constant force as c, which should produce the same motion.

3. Let

3. Let *S* represent the ſun, at an immenſe diſtance, *T* the earth (ſuppoſed, for the preſent, at reſt) *P* the moon's place in her orbit *ADBC*, in which *C*, *D*, are the quadratures, *A*, *B*, the ſyzygies: then if *PK*, parallel to *AB*, and cutting *TC* in *K*, be produced till *KL* is double of *PK*; and *LM* parallel to *PT* meet *AB* produced in *M*; *LM* and *MT* will repreſent the diſturbing forces of the ſun, by which the moon is urged in the directions *PT*, *MT*. See *Princip.* lib. i. prop. 66. and lib. iii. prop. 25, 26.

And if *TR* is made perpendicular to *LM*, the force *MT* ſhall be reſolved into two forces as *RT* and *MR*; whereof the latter, *MR*, taken from *LM*, reduces the diſturbing force, in the direction *PT*, to their difference *LR*.

4. Put now *PT* (=*LM*)=1; *PK*, the ſine of the arc *PC*=s: and then *TM* (=*PL*=3 s) : *MR* :: 1 : s; that is *MR*=$3s^2$, and *LR*, the diſturbing force in the direction *PT*, is as $1-3s^2$.

When *Cp*, the moon's diſtance from the quadrature, is an arc of 35° 15′ 52″, in which caſe $1-33^2=0$, *l* and *r* coincide; and the diſturbing force vaniſhing, the line of the *apſids* becomes ſtationary.

But if the moon's diſtance from her quadrature is ſtill greater, as at π, then $\mu\varrho$ exceeds $\mu\lambda$; and their difference $\lambda\varrho$ is a force repreſented by $-\overline{1-3s^2}$, acting in the direction Tπ. This force, at the ſyzygies, is double of *TC*.

5. Whence, and from § 1, it follows; that c being the ſun's diſturbing force, in the direction *CT*, at the quadrature; at any other point, as *P*, it will be $\pm c \times \overline{1-3s^2}$. And that writing for c the variable quantity $c \times \overline{1-3s^2}$, and *A* for the fluxion of the

 arc

arc CP, the fluent of $A \times \sqrt{\frac{1+4c \times \overline{1-3s^2}}{1+c \times \overline{1-3s^2}}}$ will give the motions of the *apsis*.

6. The quantity c being $\frac{1000}{178725}$ of the earth's mean attractive force at the moon; by computing, as above, it will be found, that while the moon moves from C to p, through an arc of 35° 15′ 52″, the total regress of the *apsis* is to the arc Cp as .005404 $(=n)$ to unity: and that the sum of its direct motions, while the moon moves from p to A, is to the arc pA as .0105707 $(=N)$ to unity.

It will be found likewise, by the inverse operation hinted in § 2, that putting $k=.00362552$, and $K=.0069611$; $+k$ and $-K$ are forces, which acting constantly, the one from C to p, the other from p to A, would produce the same motions of the *apsis*.

7. The quantities k and K might have been found, pretty near the truth, only by summing the ordinates $1LR$, or $1-3s^2$, upon the arc A: in which case we should have had $k=c \times .648869 = .00370925$, and $K=c \times 1.24018 = .006939$: and the motions thence computed would not have been much different from their just quantity. This however is mentioned, not as if the method itself were sufficiently exact; but to shew, that if, hereafter, in cases, where the limits of the forces are incomparably narrower, we shall, instead of summing the *momenta*, make use of a mean force determined in a like manner, there is no sensible error to be apprehended.

8. Hitherto we have considered the body T, round which P revolves, as quiescent; and it is thus authors have always considered it: altho' the case in nature, to

to which they meant to apply Sir Isaac Newton's rule, is widely different. The earth and moon revolve about their common centre of gravity: their distances from which being inversely as their masses, and the forces, by which either is attracted by the other, as also the forces of the sun to disturb their motions, being in the same *ratio*; it follows, that the earth in her motion round the common centre of gravity will suffer disturbances every way similar to those of the moon. And the whole motion of the *apsis* of the moon's orbit, resulting from the two disturbing forces, will be near the double of what either of them could produce separately, round a fix'd centre *.

9. To

* " *Quatenus terra et luna circum commune gravitatis centrum* " *revolvuntur, perturbabitur etiam motus terræ circa centrum illud* " *a viribus consimilibus; sed summas tam virium quam motuum re-* " *ferre licet ad lunam.*" Princip. *p.* 429.

And *p.* 141. *Apsis lunæ est duplo velocior circiter*: but this has been strangely mistaken, as if the author having revised and printed his 9th section a third time, and above forty years after it was invented, should, after all, own, that it signified nothing to his purpose. Would this be the *nil molitur inepte*, so justly applied to Newton?

See likewise, *p.* 423; where having deduced the motion of the *apsids* of Jupiter's satellites from that of our moon's, he adds, " *Diminui tamen debet motus augis sic inventus in ratione* 5 *ad* 9, " *vel* 1 *ad* 2, *circiter, ob causam, quam hic exponere non vacat.*"

The reason is not, that the orbits of Jupiter's moons are less excentric than that of ours, as some have imagined; for, " *augendo* " *vel diminuendo excentricitatem et inclinationem orbis, non mutatur* " *motus augis sensibiliter, nisi ubi eædem sunt nimis magnæ,*" p. 180. Is it not rather, because the action of the several satellites upon their primary and upon one another, in all the possible variety of directions, reduces the case of any particular satellite to that of a single body revolving round a fix'd centre, *viz.* that of Jupiter's system?

9. To determine which, we may conceive the earth as revolving in an orbit that is already in motion from the ſun's diſturbing force upon the moon: the retrograde motion of the orbit, while the earth moves from C to p, being $n \times Cp$; and the direct motion, for the reſt of the quadrant, being $N \times pA$; whence it will follow, that the diſturbing force $= k$ affects the earth's motion thro' an arc of her orbit equal to $Cp \times \overline{1+n}$; and the force $-K$ acts thro' the arc $pA \times \overline{1+N}$. And the motions of the *apſis* being in the ſame *ratio's*, if r is the regreſs of the *apſis* of the moon's orbit (determined as in § 6) and p its progreſs; the regreſs of the *apſis* of the earth's orbit will be $r \times \overline{1+n}$, and its direct motion, $p \times \overline{1-N}$. That is, the whole motions of the *apſis*, reſulting from the ſun's action upon the earth and moon together, will be ($R=$) $r \times \overline{2+n}$, and ($P=$) $p \times \overline{2-N}$; and the motions to be aſcribed to either arc, $r \times \overline{1+\frac{1}{2}n}$, and $p \times \overline{1-\frac{1}{2}N}$.

Now p, found as above, being $2082''.9$. and $N=$ $.0105707$, P is $4143''.8$. And the ſame way, $R=$ $1375''.7$: whoſe difference $P-R$ multiplied by 4, that is, $4 \times 2768''=11072''=3^{\circ}\ 4'\ 32''$, is the direct motion of the *apſis* in a revolution.

First correction for the moon's variation. Fig. 4.

10. In the foregoing calculation, it is ſuppoſed, that the moon's orbit is nearly circular, more nearly indeed than it poſſibly can be, even abſtracting from its excentricity. For altho' the moon had been projected with a direction and force to make her deſcribe

ſcribe a circle round the earth, as *EOL*, the action of the ſun would have changed this orbit into an oval, as *OADBC*; whoſe greateſt diameter, paſſing thro' the quadratures *CD*, is to the leaſt as $70\frac{1}{24}$ to $69\frac{1}{24}$. The reaſon and determination of which we have in *Princip.* lib. iii. prop. 26, 28.

11. That this action of the ſun, and the figure reſulting from it, muſt leſſen the mean motion of the apogee, is eaſily ſhewn.

For let *P* be the moon's place in her orbit, when the *apſis* is ſtationary, and *EOL* the circle of her mean motion, cutting the orbit very near the octant *O*, and *PT* in *o*: then, the accelerating forces of the earth at *P* and *o*, being inverſely as the ſquares of *PT* and *oT*, and the ſun's diſturbing force at the points *P*, *o*, being in the ſimple direct *ratio* of the ſame lines; *oT* being given, the *ratio* of the ſun's diſturbing force at the point *P*, to the earth's accelerating force at the ſame point, that is, the quantity *c* in the theorem, will be as the cube of the diſtance *PT*: and, *a fortiori*, in every point of the orbit, from the quadrature *C* to *P*, will exceed the mean force at *O*, and its effect in producing a retrograde motion of the *apſis* will be greater.

For the remaining part of the quadrant, where the motion of the *apſis* is direct, the force *c* is indeed greater than its mean quantity from *P* to *O*; but, thro' the whole octant *OA*, it is continually decreaſing as the cube of the diſtance from *T*: whence, upon the whole, that force, and its effect, from *P* to *A*, fall ſhort of their mean quantities at *O*. Seeing therefore the direct motion is diminiſhed, and the retrograde increaſed; their difference, that is, the di-

rect

rect motion in the quadrant *CPA* will be diminiſhed.

But this mean motion will be diminiſhed ſomewhat likewiſe from the inequable deſcription of the areas (in *prop.* 26. *lib.* iii.): on which account, the cubes of the diſtance *PT* muſt be every where increaſed or diminiſhed in the duplicate *ratio* of the moments of time in which a given little angle is deſcribed, to the mean moment at the octant *.

12. By computing from theſe principles, it will be found;

1. That the angle *CTP*, which was of 35° 15′ 52″ in the circle, will, in the oval orbit, be diminiſhed to 34° 43′ 34″.

2. That the *ratio* of the mean of the cubes of the moon's diſtances in the arc *CP*, to the cube of the mean diſtance, will be expreſs'd by 1.023916 ($=g$) and

* To expreſs the diſtance *PT* by *s* the ſine of the angle *CTP*, in an ellipſis not very eccentric: from any point *P* draw *PK* an ordinate to the axis *CD*, and meeting the circumſcribed circle in *M*; draw likewiſe *Mf* perpendicular to *TP* produced. Then putting $TC=1$, $TA=d$, $\frac{1-d}{d}=t$; by conjoining the *ratio's* of *TP* to *PK*, *PK* to *PM*, *PM* to *Pf*, it will be $TP=\frac{Tf}{1+ts^2}$: in which for the variable numerator *Tf*, we might, becauſe of the ſmallneſs of the angle *PTM*, write unity: but taking it rather of its mean quantity *m* (=.999987 in the moon's orbit) the diſtances, whoſe cubes are to be ſummed, will be $\frac{m}{1+ts^2}$.

And the *ratio* of the moments of time to the mean moment is that of 110,23 to $109.73+s^2$, by *prop.* 26. *lib.* iii.

and the like *ratio*, in the arc *PA*, by .9852467 (=*h*).

3. Multiplying therefore the forces *k* and —*K*, found in § 6, by *g* and by *h*, ſubſtituting the products for *c*, in the *formula*, with the arcs *CN*, and *NG*, reſpectively, and finiſhing the operation as for the circle, the regreſs, in a periodical month, will be 5548″.3, and the progreſs 16489″.8: whoſe difference is the driect mean motion ſought, 3° 21″ 2′½.

13. But nearly the ſame concluſion may be obtained, and with much leſs trouble, as follows:

In the circle *CGD*; take *CM*= 35° 15′ 52″, and thro' *P*, the point where *MK* perpendicular to *TC*, cuts the orbit, draw *TPN* meeting the circle in *N*. Then, if *R* is the regreſs of the *apſis* in a circular orbit, $R \times \overline{\frac{CM}{CN}}\big|^{\frac{1}{2}}$ will be the regreſs in the oval *CPA*.

In like manner, having inſcribed in the orbit, the circle *Amh*, and made a ſimilar conſtruction for the reſt of the quadrant $P \times \overline{\frac{Am}{Ah}}\big|^{\frac{1}{2}}$, will be the direct motion in the oval, *P* being the direct motion in a circle.

Thus, the angle of variation *MTN* being (in Dr. Halley's tables) 33′ 9″, the ſubduplicate *ratio* of *CM* to *CN* will be 1.007927, and that of *Am* to *Ah*, or of *GM* to *GN*, will be .99499. And therefore *R* (in § 9) will be augmented to 1386″.6, and *P* diminiſh'd to 4123″: whoſe difference, multiplied by 4, gives 3° 2′ 25″½; exceeding the former only by about 4″.

14. The

14. The rule is founded in this, that if, from the centre T, a circular arc Ff be deſcribed, including in the angle CTN the ſector FTf, equal to the elliptic ſector CTP, the cube of TF, the *radius* of this circle, may be taken for the mean of the cubes of the moon's diſtances in the arc CP. And becauſe the area CPT is to the ſector CMT, as PK to KM, or as TA to TC; and To or TE is a geometrical mean between TA and TC, it will eaſily appear, that $TF^3 : To^3 : : CM^{\frac{3}{2}} : CN^{\frac{3}{2}}$. And that P, found from the tables, being (nearly at leaſt) the ſtationary point in the oval, if the force k is increaſed in the ſeſquiplicate *ratio* of CM to CN, and the arc CN ſubſtituted for A in the *formula*, we ſhall, by § 1, find the retrograde motion of the *apſis*.

Now, when the conſtant force $+k$ is given, the regreſs R is as the arc A; and when A is given, and k is but a little augmented, R is proportional to k: in general therefore, if k is but a little augmented, R is as $k \times A$. Write Q for the regreſs in the oval, R ſtanding for that in the circle, already found; and it will be $Q : R : : k \times \left.\frac{CM}{CN}\right|^{\frac{3}{2}} \times CN : k \times CM$, or $Q = R \times \left.\frac{CM}{CN}\right|^{\frac{1}{2}}$, according to the rule. The like reaſoning for the direct motion.

Second correction for the Excentricity. Fig. 5.

15. This equation is inconſiderable; becauſe, altho' the *ratio* of the diſturbing force, when the moon is at a greater than her mean diſtance, is more increaſed than it is diminiſhed in the oppoſite points of her orbit;

orbit; this increaſe is very near compenſated by the comparative ſmallneſs of the angular velocity.

Let *ADa* repreſent the moon's elliptic orbit, whoſe centre is *C*, its axes *Aa*, *Dd*, the mean excentricity *CT*, and the circle of her mean motion *MDmd*, cutting *Aa* in *M* and *m*. Then, becauſe it is a mean motion we ſeek, generated while the axis *Aa* paſſes thro' all its different aſpects of the ſun; we may conceive the direct motion already found, of $3^{\circ}\ 2'\ 21''\frac{1}{2}$, to be produced by a conſtant diſturbing force $-\dot{K}$, acting on the moon as ſhe revolves in her circular orbit *MDmd*; and we have only to enquire, how much this force, and its effects, are to be increaſed, the moon really moving about the ſame centre *T*, in the elliptic arc *AD*; and how much diminiſhed in the arc *Da*.

16. For which purpoſe, the conſtant force $\dot{K}$ is to be increaſed in the *ratio* of the mean of the cubes of the moon's diſtances, in the arc *AD*, to the cube of *TD* or *CA*, and diminiſhed as the mean of the cubes of the diſtances in *Da*. Let the forces reſulting be $\dot{K} \times G$ and $\dot{K} \times H$; and theſe being ſubſtituted in the *formula*, with the arcs 2*DM*, 2*Dm*, reſpectively, the ſum of the motions found will be the whole mean motion of the apogee, including the correction for the excentricity.

Now $\dot{K}$ will be found to be .00557337, and the excentricity *TC* being .05505, and *Q* the quadrantal arc to *radius* 1; the *ratio* *G*, or, which is the ſame, the ſeſquiplicate of the time, in which the elliptic arc *AD* is deſcribed, to the time in the circular arc *DM*, that is, $\overline{\frac{Q+TC}{DM}}\Big|^{\frac{3}{2}}$ will be 1.110942; and *H* $\left(=\overline{\frac{Q-TC}{DM}}\Big|^{\frac{3}{2}}\right.$

K

=.9001387: whence the whole motion, found as above directed, will be $10962'' = 3^{\circ}\ 2'\ 42''$; the correction, on account of the excentricity, being only $21''$.

Multiply $3^{\circ}\ 2'\ 42''$ by 1.080853, and the product $3^{\circ}\ 17'\ 28''$ is the mean motion of the apogee, in a ſynodical month; exceeding the quantity marked in the tables by no more than $4''$.

17. Of the obliquity of the moon's orbit, to the plane of the ecliptic, we take no notice: becauſe, altho', abſolutely ſpeaking, a force in that plane, referred to the moon's orbit, would, thence, be diminiſh'd by about $\frac{3}{1000}$ parts; yet, in the preſent caſe, the effect of the obliquity is included in the firſt determination of the quantity *c*, from the periodical times of the earth and moon; all but what belongs to the corrections; and which is only $110''.\times 003 = 0''.33$, to be ſubtracted.

18. The force *c* is, itſelf, the effect of the ſun's parallax, and the total effect; excepting only a ſmall difference between his action on the moon, when ſhe is waxing or waning, and when ſhe is in the other half of her orbit; neglected as altogether inconſiderable.

Upon the whole, we may conclude, that, in this, as in the other phænomena of the celeſtial motions, the principles and rules of Sir Iſaac Newton are fully confirmed and verified.

Fig. 3. p. 64.

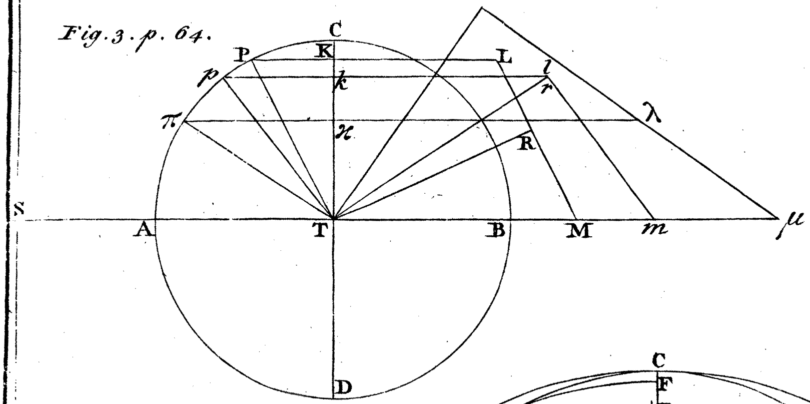
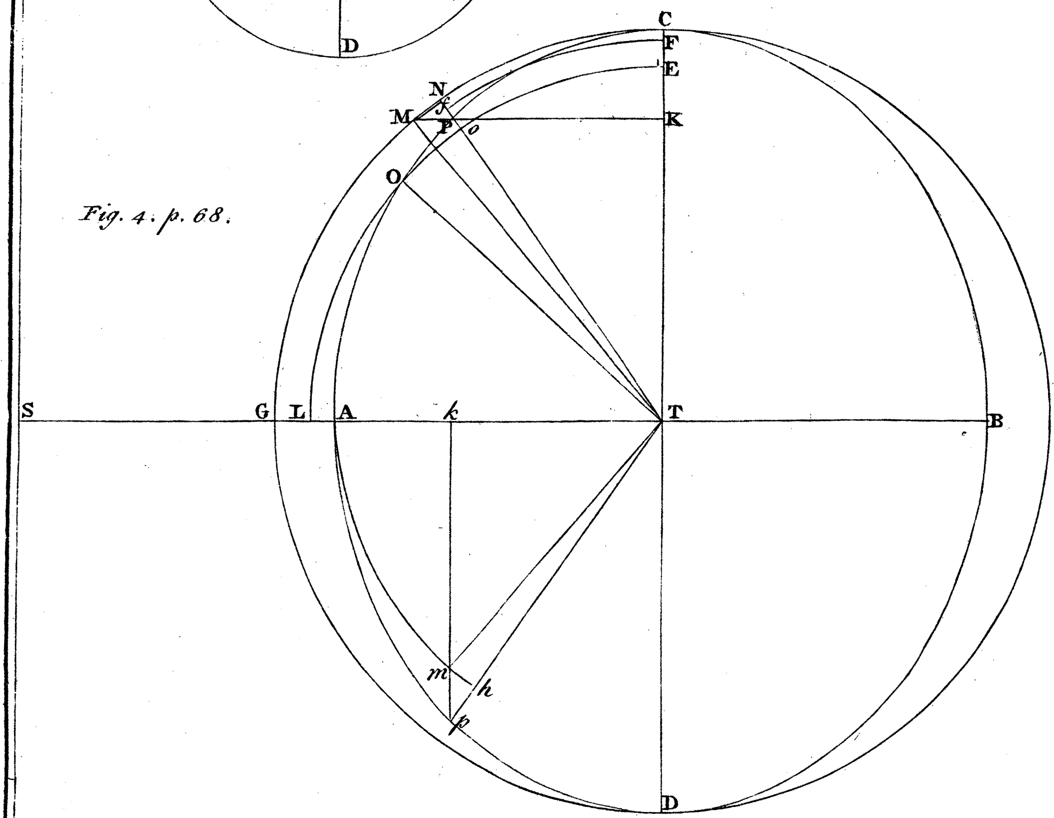
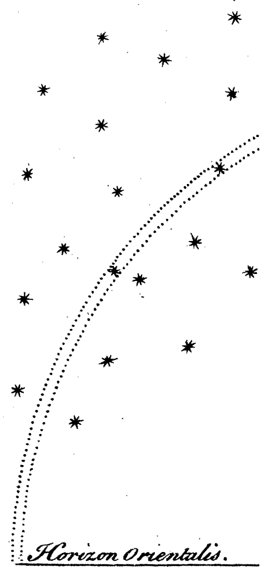


Fig. 4. p. 68.





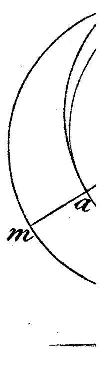


Fig. 2. p. 39.

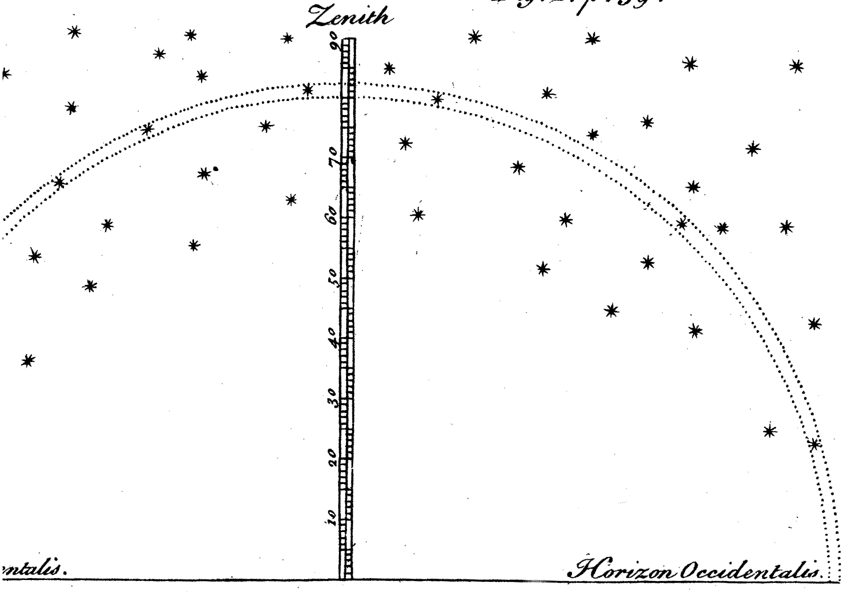


Fig. 5. p. 73.

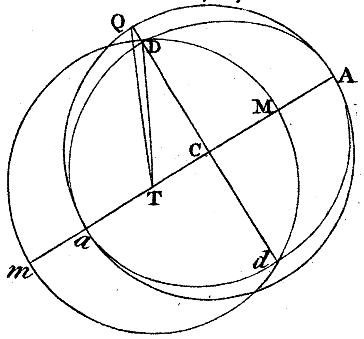
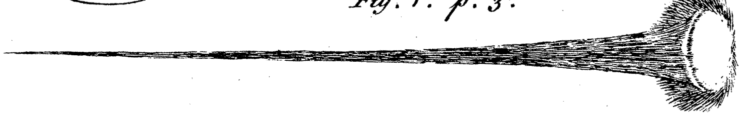
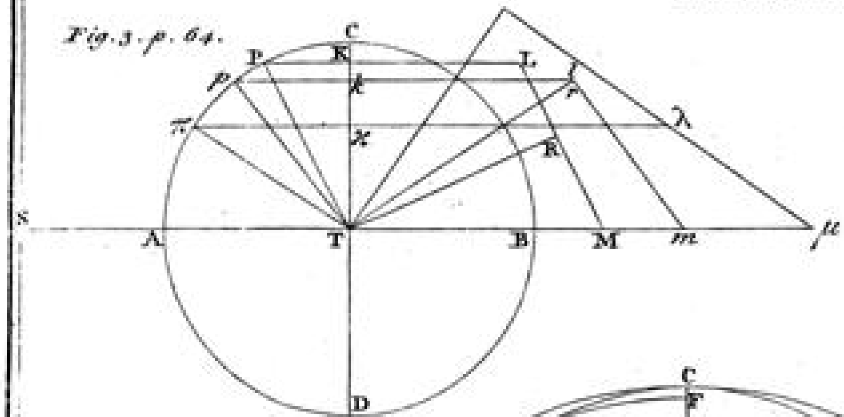


Fig. 1. p. 3.





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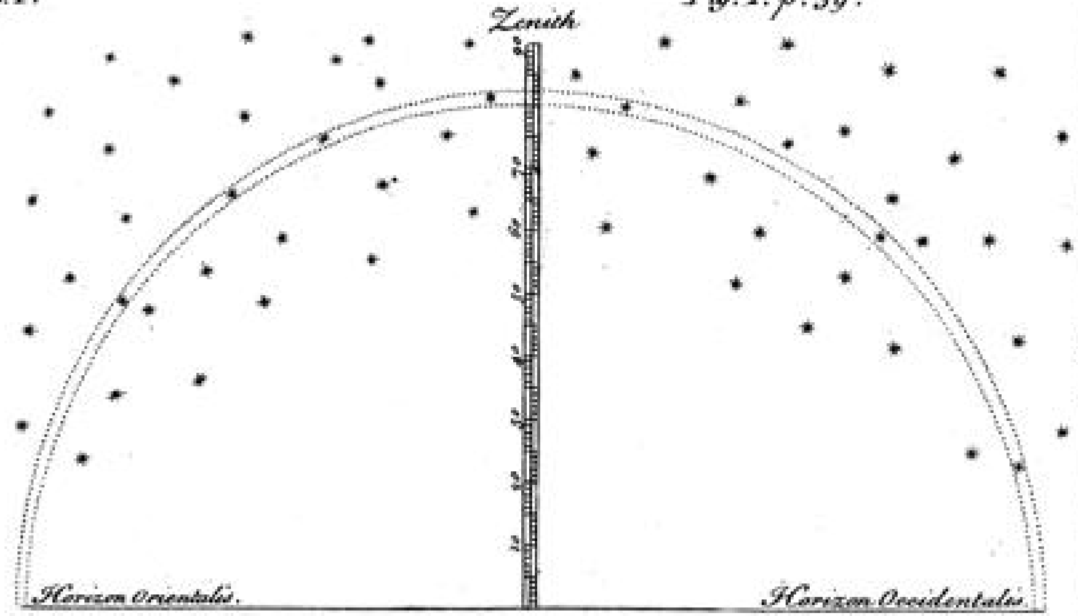


Fig. 2. p. 39.

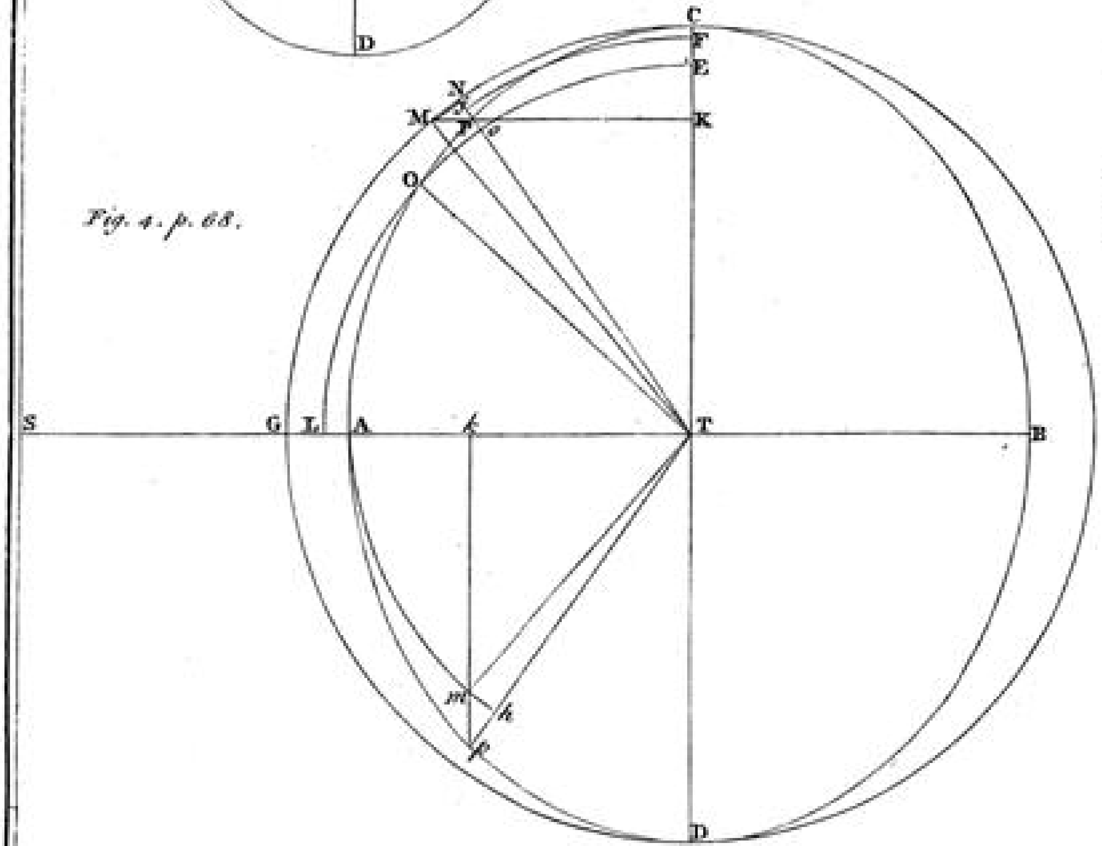
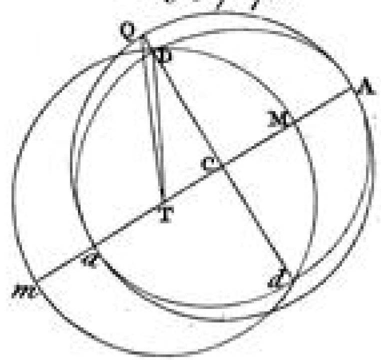


Fig. 4. p. 68.



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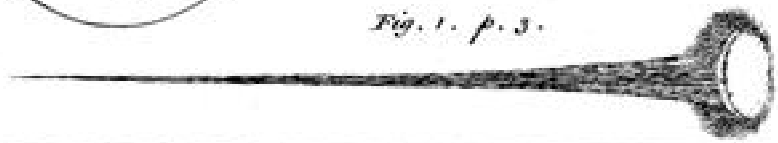

Fig. 1. p. 3.